The Impact of Estimation Error in Portfolio Optimization

Thesis presented in partial satisfaction of the requirements to obtain the degree of Maestro en Administración by:

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Full-Time MBA Program 55

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The impact of estimation error in portfolio optimization

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Executive Summary:

The objective of this paper is to better understand the impact that estimation error in the parameters has on portfolio performance and to identify ways to reduce it. To do so, we worked within the frame of Modern Portfolio Theory. Then, several portfolio rules have been applied to 14 data sets. They were analyzed through calculation experiments using MATLAB software. The ability of the various portfolio rules to reduce the impact of estimation error depending on several variables were measured and understood. The study shows that adding constraints to the portfolios is an effective way to mitigate the impact of estimation error. This may allow constrained portfolios to achieve greater expected utility compared to the tangency portfolio, but equally weighted portfolios remain the best way to build a portfolio when little information is available or when the portfolio is composed of many assets.

Resumen Ejecutivo:

El objetivo de este trabajo es comprender mejor el impacto que el error de estimación en los parámetros tiene sobre el rendimiento de un portafolio e identificar formas de reducirlo. Para eso, trabajamos en el marco de la Teoría Moderna del Portafolio. A partir de esto, se han aplicado y analizado varias reglas de portafolio a 14 conjuntos de datos. Luego, se analizaron mediante experimentos de cálculo utilizando el software de MATLAB. Se midió y entendió la capacidad de las diversas reglas de la cartera para reducir el impacto del error de estimación en función de varias variables. El estudio muestra que añadir restricciones a los portafolios es una forma eficaz de mitigar el impacto del error de estimación. Eso puede permitir que los portafolios restringidos logren una mayor utilidad esperada en comparación con el portafolio tangente, pero el portafolio de pesos iguales sigue siendo la mejor manera de crear un portafolio cuando se dispone de poca información o que el portafolio está compuesto por muchos activos.

1. INTRODUCTION

Conceptualized in 1952 by Harry Markowitz, Modern Portfolio Theory (MPT) is based on the principle that an optimal portfolio is the combination of the riskless asset (F) and the tangency portfolio (T) composed solely of risky assets. Even though the proportions between assets (F) and (T) are up to the investor –depending on his risk profile–, the main idea is that in any case he can only invest in 2 types of assets: those composing the tangency portfolio and the riskless one. Such an optimal portfolio is called a two-fund portfolio rule.

As stated by the MPT, the goal of the investor is to maximize his utility through a mean-variance (MV) analysis. His objective is to allocate his resources to build an optimal portfolio: one that has the highest expected return (mean) for a given level of risk, or equivalently, one that has the lowest risk (variance) for a given expected return.

1.1 Portfolio Optimization Problem: Theory

An investor is building a portfolio using a riskless asset (F) and *n* risky assets. The rates of return on these assets at time *t* are called $\boldsymbol{r_{ft}}$ and $\boldsymbol{r_t}$, respectively. It is assumed that $t > n$ and excess returns are defined as $\boldsymbol{R_t} = \boldsymbol{r_t} - \boldsymbol{r_{ft}}\,\mathbf{1}$, where $\mathbf{1}$ is a *n*-vector of unit entries. Concerning its probability distribution, we assume that $\boldsymbol{R_t}$ is independent and identically distributed (i.i.d) over time. In addition, it is assumed that $\boldsymbol{R_t}$ follows a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

Given the portfolio weights $\boldsymbol{w}_p$, an *n*-vector on the risky assets, the excess return on the portfolio at time *t* is $R_{pt} = \boldsymbol{w}_p{}'\boldsymbol{R_t}$, whose mean and variance are given by $\mu_p = \boldsymbol{w}_p{}'\boldsymbol{\mu}$ and $\sigma_p^2 = \boldsymbol{w}_p{}'\boldsymbol{\Sigma}\boldsymbol{w}_p$. The parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ will be referred as the “true parameters”. The mean-variance utility of a portfolio p is given by

$$U(\boldsymbol{w}_p) = \mu_p - \frac{\gamma}{2}\,\sigma_p^2, \tag{1}$$

where γ represents the coefficient of risk aversion (risk profile) of the investor, which satisfies $0 < \gamma < \infty$. The goal of the investor is to solve the following portfolio optimization problem (MV):

$$\text{Maximize } U(\boldsymbol{w}_p) = \boldsymbol{w}_p{}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}_p{}'\boldsymbol{\Sigma}\boldsymbol{w}_p. \tag{2}$$

When there is no parameter uncertainty (i.e. no estimation error) the investor knows $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$; in this situation he will build an optimal tangent portfolio (T) so that

$$\boldsymbol{w}_T = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{3}$$

The resulting expected utility of this solution is

$$U(\boldsymbol{w}_T) = \frac{1}{2\gamma}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \frac{\theta_T^2}{2\gamma}, \tag{4}$$

where $\theta_T^2 = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ is the squared Sharpe ratio of the tangency portfolio (T). For a given risk aversion parameter γ, $U(\boldsymbol{w}_T)$ is the highest –theoretical– utility that an investor's portfolio can reach.

1.2 Portfolio Optimization Problem: Reality, ML estimation method

In real life, the investor neither knows $\boldsymbol{\mu}$ nor $\boldsymbol{\Sigma}$, so if he were to build a portfolio for the period *t+1*, he would first have to estimate these parameters. Using historical data, the investor can use the maximum likelihood (ML) estimation method. We note that better estimators exist, such as the unbiased ones, but that is not the theme of this paper.

Let Φ_t be *t* monthly periods of observed returns data so that $\Phi_t = \{\boldsymbol{R_1}, \boldsymbol{R_2}, \dots, \boldsymbol{R_t}\}$. Based on this, the investor can now calculate the sample mean and covariance matrix $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$ respectively defined as

$$\widehat{\boldsymbol{\mu}} = \frac{1}{\mathrm{t}}\sum_{i=1}^{t}\boldsymbol{R_i}, \tag{5}$$

and

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{t}\sum_{i=1}^{t}(\boldsymbol{R_i} - \widehat{\boldsymbol{\mu}})(\boldsymbol{R_i} - \widehat{\boldsymbol{\mu}})'. \tag{6}$$

Statistically, these are the maximum likelihood estimators of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. This means that by plugging-in these estimators into the original portfolio weights formula, the investor can now calculate $\widehat{\boldsymbol{w}}_T$, the maximum likelihood estimator of the unknown portfolio weight vector $\boldsymbol{w}_T$. We replace $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in (3) by $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$ so that

$$\widehat{\boldsymbol{w}}_T = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\mu}}. \tag{7}$$

The out-of-sample performance of such a plug-in portfolio $\widehat{\boldsymbol{w}}$ is given by

$$U(\widehat{\boldsymbol{w}}_T) = \widehat{\boldsymbol{w}}_T{}'\boldsymbol{\mu} - \frac{\gamma}{2}\,\widehat{\boldsymbol{w}}_T{}'\boldsymbol{\Sigma}\widehat{\boldsymbol{w}}_T \tag{8}$$

1.3 Introducing the concept of estimation error

Using the vector $\boldsymbol{w}_T$ in (3) allows the creation of an optimal portfolio, leading to the highest possible utility $U(\boldsymbol{w}_T)$. But as explained earlier, the parameters used to calculate $\boldsymbol{w}_T$ are unknown and must be estimated, leaving us with $\widehat{\boldsymbol{w}}_T$ in (7), a plug-in estimator of $\boldsymbol{w}_T$.

For any portfolio p, using an estimator $\widehat{\boldsymbol{w}}_p$ instead of $\boldsymbol{w}_p$ always comes with unavoidable estimation error, which has a negative impact on the performance of the portfolio. Estimation error arises from the uncertainty in the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$; it is the difference between the optimal portfolio weights vector $\boldsymbol{w}_p$ and its estimator $\widehat{\boldsymbol{w}}_p$. It causes the investor to not optimally invest his resources into the different risky assets. If we knew the true parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, there would not be any estimation error. Estimation error has a negative impact on the performance of the portfolio since

$$U(\boldsymbol{w}_p) - U(\widehat{\boldsymbol{w}}_p) > 0. \tag{9}$$

The loss function caused by using $\widehat{\boldsymbol{w}}_{\boldsymbol{p}}$ instead of $\boldsymbol{w}_{\boldsymbol{p}}$ is defined as

$$L(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p) = U(\boldsymbol{w}_p) - U(\widehat{\boldsymbol{w}}_p). \tag{10}$$

From there, we have the expected loss function that is given by

$$\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p) = E\left[L(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)\right] = U(\boldsymbol{w}_p) - E\left[U(\widehat{\boldsymbol{w}}_p)\right]. \tag{11}$$

2. PORTFOLIO RULES AND IMPACT OF ESTIMATION ERROR ON THEM

2.1 The issue of estimation error under the classic two-fund portfolio rule

Kan and Zhou (2007) have studied the loss function associated with the use of estimators rather than the use of the true parameters. They have showed that, for *t* observations and *n* assets:

a) When the covariance matrix is known,

$$\mathbb{E}[U(\widehat{\boldsymbol{w}}_T)|\boldsymbol{\Sigma}] = \frac{\theta_T^2}{2\gamma} - \frac{n}{2\gamma t}, \tag{12}$$

leading to the performance loss

$$\rho(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\Sigma}) = U(\boldsymbol{w}_T) - \mathbb{E}[U(\widehat{\boldsymbol{w}}_T)|\boldsymbol{\Sigma}] = \frac{n}{2\gamma t}. \tag{13}$$

b) When the mean is known,

$$\mathbb{E}[U(\widehat{\boldsymbol{w}}_T)|\boldsymbol{\mu}] = k_0 \frac{\theta_T^2}{2\gamma} \tag{14}$$

and

$$\rho(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\mu}) = (1-k_0)\frac{\theta_T^2}{2\gamma} \tag{15}$$

where

$$k_0 = \left(\frac{t}{t-n-2}\right)\left[2 - \frac{t(t-2)}{(t-n-1)(t-n-4)}\right]. \tag{16}$$

c) When both the covariance matrix $\boldsymbol{\Sigma}$ and the mean $\boldsymbol{\mu}$ are to be estimated using the ML estimation method,

$$\mathbb{E}[U(\widehat{\boldsymbol{w}}_T)] = k_0 \frac{\theta_T^2}{2\gamma} - \frac{nt(t-2)}{2\gamma(\mathrm{t}-\mathrm{n}-1)(\mathrm{t}-\mathrm{n}-2)(\mathrm{t}-\mathrm{n}-4)} \tag{17}$$

and

$$\rho(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T) = (1-k_0)\frac{\theta_T^2}{2\gamma} + \frac{nt(t-2)}{2y(t-n-1)(t-n-2)(t-n-4)}. \tag{18}$$

2.2 The orthogonal three-fund portfolio rule and its estimation error

2.2.1 Introducing the orthogonal three-fund portfolio rule

As proposed by Kan and Zhou (2007), in the presence of estimation error, a solution to mitigate the performance loss is to allocate a portion of the investment resources into the minimum-variance portfolio (G), added to the tangency portfolio (T) and the risk-free asset (F). Such a portfolio is called a three-fund rule portfolio.

In addition to that, Chávez-Bedoya and Rosales (2019) showed that the performance loss mitigation resulting in the use of a three-fund portfolio is due to the degree of orthogonality of its components. They introduced a three-fund portfolio that mixes F, H and G, where H is a maximum performance zero-investment portfolio that is orthogonal to G. This means that its objective is to maximize utility, that the value of the sum of its assets is equal to zero and that there is no covariance between portfolio H and portfolio G. The minimum-variance portfolio G is defined by

$$\boldsymbol{w}_G = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \,, \tag{19}$$

leading to the performance

$$U(\boldsymbol{w}_G) = \mu_G - \frac{\gamma}{2}\sigma_G^2. \tag{20}$$

The zero-investment portfolio H is defined by

$$\boldsymbol{w}_H = \frac{1}{\gamma}\boldsymbol{R\mu} \tag{21}$$

where

$$\boldsymbol{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \,. \tag{22}$$

The zero-investment portfolio (H) is also called "Hedge portfolio." Its performance measures are given by

$$U(\boldsymbol{w}_H) = \frac{\psi^2}{2\gamma}; \quad \psi^2 = \boldsymbol{\mu}'\boldsymbol{R\mu} \tag{23}$$

where ψ^2 is the squared Sharpe ratio of portfolio H.

Now that the portfolios G and H have been introduced, we can present the portfolio Q, which is the sum of the latter two. As G and H are orthogonal, we have

$$\boldsymbol{w}_G'\boldsymbol{\Sigma}\boldsymbol{w}_H = 0, \tag{24}$$

meaning that the returns of G and H are uncorrelated so that

$$\boldsymbol{w}_Q = \boldsymbol{w}_G + \boldsymbol{w}_H; \quad U\left(\boldsymbol{w}_Q\right) = U(\boldsymbol{w}_G) + U(\boldsymbol{w}_H). \tag{25}$$

Using the ML estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, the plug-in estimators of portfolios G and H are

$$\widehat{\boldsymbol{w}}_G = \frac{\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}; \quad \widehat{\boldsymbol{w}}_H = \frac{1}{\gamma}\widehat{\boldsymbol{R}}\widehat{\boldsymbol{\mu}}, \tag{26}$$

where

$$\widehat{\boldsymbol{R}} = \frac{\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}.$$

Similarly, portfolio rule $\widehat{\boldsymbol{w}}_Q$ equals to the sum of the plug-in estimators of portfolios G and H so that

$$\widehat{\boldsymbol{w}}_Q = \widehat{\boldsymbol{w}}_G + \widehat{\boldsymbol{w}}_H; \quad U\left(\widehat{\boldsymbol{w}}_Q\right) = U(\widehat{\boldsymbol{w}}_G) + U(\widehat{\boldsymbol{w}}_H). \tag{27}$$

2.2.2. The issue of estimation error under the orthogonal three-fund portfolio

Now that the orthogonal three-fund portfolio has been introduced, we can focus on its expected out-of-sample performance and expected loss, given by:

a) When $\boldsymbol{\Sigma}$ is known:

$$\mathbb{E}\left[U\left(\widehat{\boldsymbol{w}}_Q\right)|\boldsymbol{\Sigma}\right] = \mu_G - \frac{\gamma}{2}\sigma_G^2 + \frac{\psi^2}{2\gamma} - \frac{n-1}{2\gamma t}, \tag{28}$$

$$\rho\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q \middle| \boldsymbol{\Sigma}\right) = \frac{n-1}{2\gamma t}. \tag{29}$$

b) When $\boldsymbol{\mu}$ is known:

$$\mathbb{E}\left[U\left(\widehat{w}_Q\right)|\boldsymbol{\mu}\right] = \mu_G - \frac{\gamma}{2}\left(\frac{t-2}{t-n-1}\right)\sigma_G^2 + k_1\frac{\psi^2}{2\gamma}, \tag{30}$$

$$\rho\left(w_Q, \widehat{w}_Q \middle| \boldsymbol{\mu}\right) = \frac{\gamma}{2}\left(\frac{n-1}{t-n-1}\right)\sigma_G^2 + (1-k_1)\frac{\psi^2}{2\gamma}, \tag{31}$$

where

$$k_1 = \left(\frac{t}{t-n-1}\right)\left[2 - \frac{t(t-2)}{(t-n)(t-n-3)}\right]. \tag{32}$$

c) When both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown and estimated using the ML estimation method:

$$\mathbb{E}\left[U\left(\widehat{\boldsymbol{w}}_Q\right)\right] = \mu_G - \frac{\gamma}{2}\left(\frac{t-2}{t-n-1}\right)\sigma_G^2 + k_1\frac{\psi^2}{2\gamma} - k_2\frac{1}{2\gamma}, \tag{33}$$

$$\rho\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q\right) = \frac{\gamma}{2}\left(\frac{n-1}{t-n-1}\right)\sigma_G^2 + (1-k_1)\frac{\psi^2}{2\gamma} + k_2\frac{1}{2\gamma}, \tag{34}$$

where

$$k_2 = \frac{(n-1)t(t-2)}{(t-n)(t-n-1)(t-n-3)}. \tag{35}$$

2.3 The constrained three-fund portfolio rule and its estimation error

2.3.1 Introducing the idea of constraints

Let **A** be an *m* x *n* full row rank matrix with $m < n$ and $\boldsymbol{b} \neq \mathbf{0}$ be an *m*-vector so that the augmented matrix [**A b**] has rank *m* and assumes $\boldsymbol{A\Sigma}^{-1}\boldsymbol{\mu} \neq \mathbf{0}$. For the optimal MV portfolio to satisfy a set of *m* linear constraints given by $\boldsymbol{Aw} = \boldsymbol{b}$, we need to solve the following optimization problem (MV2):

$$\begin{aligned}\boldsymbol{w}_Q &= argmax_{\boldsymbol{w}}\{U(\boldsymbol{w}) \mid \boldsymbol{Aw} = \boldsymbol{b}\} \\ &= argmax_{\boldsymbol{w}}\left\{\boldsymbol{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}'\boldsymbol{\Sigma w}\middle|\boldsymbol{Aw} = \boldsymbol{b}\right\} \\ &= \boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\boldsymbol{A\Sigma}^{-1}\boldsymbol{A}')^{-1}\boldsymbol{b} + \frac{1}{\gamma}\boldsymbol{R\mu} \qquad (36)\end{aligned}$$

where the matrix **R** is given by

$$\boldsymbol{R} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\boldsymbol{A\Sigma}^{-1}\boldsymbol{A}')^{-1}\boldsymbol{A\Sigma}^{-1}. \qquad (37)$$

The portfolio weights vector $\boldsymbol{w}_Q$, the solution to MV2, is used to build portfolio Q, called a constrained three-fund rule.

2.3.2 Relation between orthogonal three-fund rule portfolio and constrained portfolio

It is interesting to note that the orthogonal three-fund rule portfolio Q presented in Section *2.2.1* is the solution to a specific case of the MV2 optimization problem in which *m*=1 and **b**=1; there, **A** is a *1* x *n* all-ones matrix. That is why from now on portfolio Q will be the name of the solution to MV2 showed in equation (36).

Portfolio Q, formerly presented in Section *2.2.1,* is a particular case and can be expressed in a more general way. As stated earlier, Q is the sum of a minimum-variance portfolio G and a maximum performance portfolio H. The portfolios G and H also have been introduced earlier, but here again we can show their expressions in a more general way:

Portfolio G is the solution to the following optimization problem:

$$\begin{aligned}\boldsymbol{w}_G &= \text{argmin}_{\boldsymbol{w}}\{\sigma^2 \mid \boldsymbol{Aw} = \boldsymbol{b}\} \\ &= \text{argmin}_{\boldsymbol{w}}\{\boldsymbol{w}'\boldsymbol{\Sigma w} | \boldsymbol{Aw} = \boldsymbol{b}\} \\ &= \boldsymbol{\Sigma}^{-1}\boldsymbol{A}'(\boldsymbol{A\Sigma}^{-1}\boldsymbol{A}')^{-1}\boldsymbol{b}. \qquad (38)\end{aligned}$$

Portfolio H is the solution to the following optimization problem:

$$\begin{aligned}\boldsymbol{w}_H &= \text{argmax}_{\boldsymbol{w}}\{U(\boldsymbol{w})|\mathbf{Aw}=\mathbf{0}\}\\ &= \text{argmax}_{\boldsymbol{w}}\left\{\boldsymbol{w}'\boldsymbol{\mu}-\frac{\gamma}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}\middle|\mathbf{Aw}=\mathbf{0}\right\}\\ &= \frac{1}{\gamma}\boldsymbol{R}\boldsymbol{\mu}.\end{aligned} \tag{39}$$

For any vector $\mathbf{b}$, $\boldsymbol{w}_G'\boldsymbol{\Sigma}\boldsymbol{w}_H = 0$, which means that the portfolios G and H are orthogonal. Consequently, portfolio Q is the sum of the portfolios G and H and its performance is the sum of the individual performances as showed in equation (25). From now and throughout the remainder of this paper, we respectively replace the specific cases (19) and (21) by their more general expression showed in the equations (38) and (39).

With the ML estimation method, we use the sample mean $\widehat{\boldsymbol{\mu}}$ and covariance matrix $\widehat{\boldsymbol{\Sigma}}$ to get a plug-in estimator of $\boldsymbol{w}_Q$:

$$\widehat{\boldsymbol{w}}_Q = \widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{A}'\left(\boldsymbol{A}\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{A}'\right)^{-1}\boldsymbol{b}+\frac{1}{\gamma}\widehat{\boldsymbol{R}}\widehat{\boldsymbol{\mu}}, \tag{40}$$

where

$$\widehat{\boldsymbol{R}} = \widehat{\boldsymbol{\Sigma}}^{-1}-\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{A}'\left(\boldsymbol{A}\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{A}'\right)^{-1}\boldsymbol{A}\widehat{\boldsymbol{\Sigma}}^{-1} \tag{41}$$

is the estimator of matrix $\mathbf{R}$ in (37).

2.3.3. The issue of estimation error under the constrained portfolios

Now that the notion of constrained portfolios has been introduced and that its relationship with the orthogonal portfolios is clearly established, we can focus on the expected out-of-sample performance and expected loss functions of the constrained portfolio Q.

a) When $\boldsymbol{\Sigma}$ is known:

$$\mathbb{E}\left[U(\widehat{\boldsymbol{w}}_Q)|\boldsymbol{\Sigma}\right] = \mu_G-\frac{\gamma}{2}\sigma_G^2+\frac{\psi^2}{2\gamma}-\frac{n-m}{2\gamma t}, \tag{42}$$

$$\rho\left(\mathbf{w}_{\mathrm{Q}},\widehat{\mathbf{w}}_{\mathrm{Q}}|\boldsymbol{\Sigma}\right) = \frac{n-m}{2\gamma t}. \tag{43}$$

b) When $\boldsymbol{\mu}$ is known:

$$\mathbb{E}\left[U(\widehat{\boldsymbol{w}}_Q)|\boldsymbol{\mu}\right] = \mu_G-\frac{\gamma}{2}\left(\frac{t-2}{t-n+m-2}\right)\sigma_G^2+c_1\frac{\psi^2}{2\gamma}, \tag{44}$$

$$\rho(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q|\boldsymbol{\mu}) = \frac{\gamma}{2}\left(\frac{n-m}{t-n+m-2}\right)\sigma_G^2 + (1-c_1)\frac{\psi^2}{2\gamma}, \tag{45}$$

where

$$c_1 = \left(\frac{t}{t-n+m-2}\right)\left[2 - \frac{t(t-2)}{(t-n+m-1)(t-n+m-4)}\right]. \tag{46}$$

c) When both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown and must be estimated:

$$\mathbb{E}[U(\widehat{\boldsymbol{w}}_Q)] = \mu_G - \frac{\gamma}{2}\left(\frac{t-2}{t-n+m-2}\right)\sigma_G^2 + c_1\frac{\psi^2}{2\gamma} - c_2\frac{1}{2\gamma}, \tag{47}$$

$$\rho(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q) = \frac{\gamma}{2}\left(\frac{n-1}{t-n+m-2}\right)\sigma_G^2 + (1-c_1)\frac{\psi^2}{2\gamma} + c_2\frac{1}{2\gamma}, \tag{48}$$

where

$$c_2 = \frac{(n-m)t(t-2)}{(t-n+m-1)(t-n+m-2)(t-n+m-4)}. \tag{49}$$

Note: for the reasons explained in Section *2.3.2* and to avoid redundancies, we now respectively replace the specific cases (28), (29), (30), (31), (33) and (34) by their more general expressions (42), (43), (44), (45), (47) and (48). This means that the findings made for the constrained portfolios also apply to the orthogonal portfolio, as the latter is only a specific case of the former.

3. PERFORMANCE-LOSS OF THE PORTFOLIO RULES

3.1 Expressions used to calculate the expected absolute and relative performance loss

When the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown, the expression of the expected absolute loss is given by (11). The expected absolute loss of a portfolio p as a function of which parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are known is given by the following expressions.

a) When $\boldsymbol{\Sigma}$ is known, meaning that the loss is due to the use of $\widehat{\boldsymbol{\mu}}$ instead of $\boldsymbol{\mu}$, the expression of the expected absolute loss is:

$$\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\Sigma}) = E[L(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\Sigma})] = U(\boldsymbol{w}_p) - E[U(\widehat{\boldsymbol{w}}_p|\boldsymbol{\Sigma})]. \tag{50}$$

b) When $\boldsymbol{\mu}$ is known, meaning that the loss is due to the use of $\widehat{\boldsymbol{\Sigma}}$ instead of $\boldsymbol{\Sigma}$, the expression of the expected absolute loss is:

$$\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\mu}) = E[L(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\mu})] = U(\boldsymbol{w}_p) - E[U(\widehat{\boldsymbol{w}}_p|\boldsymbol{\mu})]. \tag{51}$$

The expressions of the expected relative loss of a portfolio p as a function of which parameters are known are given by:

$$\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p \middle| \boldsymbol{\Sigma}\right) = \frac{\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p \middle| \boldsymbol{\Sigma}\right)}{U\left(\boldsymbol{w}_p\right)} * 100, \tag{52}$$

$$\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p \middle| \boldsymbol{\mu}\right) = \frac{\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p \middle| \boldsymbol{\mu}\right)}{U\left(\boldsymbol{w}_p\right)} * 100, \tag{53}$$

$$\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right) = \frac{\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)}{U\left(\boldsymbol{w}_p\right)} * 100. \tag{54}$$

These are the percentage loss expressions of the expected-out-of-sample performances from holding portfolios whose parameters have been estimated –through ML estimators– instead of using the true parameters. These expressions are important because they are the tools that we will use through the document to calculate the expected utility of the portfolios and to compare their performances.

$$\mathbb{E}\left[U\left(\widehat{\boldsymbol{w}}_p\right)\right] = U\left(\boldsymbol{w}_p\right) * \left(1 - \%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)\right) \tag{55}$$

The expected utility of a portfolio p is given by expression (55). This depends on the maximum utility of the portfolio and its expected relative loss. The expected utility is what ultimately defines the performance of a portfolio.

3.2 General observations and considerations for the investor

In this section, we describe the influence of different variables on estimation error and their impact on the performance of the portfolio rules presented above. The studied variables are: historical return periods, Sharpe ratio and number of assets in the portfolio. For comparability of the results, the coefficient of risk aversion will be held constant throughout the document with γ=3. Whenever we make a statement about a variable, it is implied that it is for “all other things being equal.” Unless otherwise stated, the following findings apply to all the portfolios introduced earlier, hence the use of p in the expressions.

3.2.1 Periods of historical return

Including more periods of historical returns (greater *t*) to calculate the ML estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ –as described in (5) and (6)– allows those estimators to be closer to the true parameters. Thus, the larger the sample size *t*, the smaller the estimation error, resulting in a reduction in $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$. This means that an investor should gather as much data as possible concerning the assets he wants to invest in.

3.2.2 Sharpe ratio of the risky-assets portfolios

Throughout this paper, whenever it comes to the Sharpe ratio of the constrained portfolios Q, we will use ψ –which is actually the Sharpe ratio of portfolio H, part of Q – because it is the main driver of the actual constrained portfolios' Sharpe ratio.

a) Loss due to the use of $\widehat{\boldsymbol{\mu}}$.

The expected absolute loss due to the use of $\widehat{\boldsymbol{\mu}}$ instead of $\boldsymbol{\mu}$ –$\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\Sigma})$– is not affected by the Sharpe ratio of the risky-assets portfolio. It is constant and independent from it, as shown respectively in the equations (29) and (43) for the tangent and the constrained portfolios. But as an increase in the Sharpe ratio of the risky-assets portfolio leads to a higher maximum utility, as shown by (4), this also leads to a reduction in the relative loss $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\Sigma})$.

b) Loss due to the use of $\widehat{\boldsymbol{\Sigma}}$

Ceteris paribus, we can see in the equations (15) and (45) –for the tangent and the constrained portfolios, respectively– that the higher the Sharpe ratio of a portfolio is, the higher $\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\mu})$ will be. The latter expression is the expected absolute loss due to the use of $\widehat{\boldsymbol{\Sigma}}$ instead of the true parameter $\boldsymbol{\Sigma}$. As shown in (14) and (17) for the tangent portfolios and in (45) and (48) for the constrained ones, for a given increase in the Sharpe ratio, $\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p|\boldsymbol{\mu})$ increases parallelly to the expected absolute loss due to estimation error for both parameters, expressed by $\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$. However, for the tangent portfolio, the expected relative loss due to the use of $\widehat{\boldsymbol{\Sigma}}$ –expressed by $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\mu})$– is constant no matter the Sharpe ratio. This can be explained by the fact that an increase in the Sharpe ratio leads to a higher maximum utility, offsetting the higher expected absolute loss; hence the unchanged expected relative loss, which is the relation between the expected absolute loss and the maximum utility of the portfolio as shown in (53).

In the case of the constrained portfolios, $\%L_Q\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q \middle| \boldsymbol{\mu}\right)$ decreases when the Sharpe ratio is higher, meaning that the increased maximum utility is higher than the increment of $\rho\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q \middle| \boldsymbol{\mu}\right)$. The orthogonality and the use of the minimum-variance portfolio in the constrained portfolios allow them, by design, to have a lower estimation error in the covariance matrix $\boldsymbol{\Sigma}$ compared to the tangent portfolio, thus reducing the performance loss due to the use of $\widehat{\boldsymbol{\Sigma}}$. We can also note that the Sharpe ratio of a constrained portfolio is always lower than the one of an equivalent (with the same set of assets) tangent portfolio.

c) Loss solely due to the interactive effect from using both estimators $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$

Similarly, we can see that the expected absolute loss solely due to the interaction from using both estimators is constant and independent from the Sharpe ratio. This is shown by deducting (29) and (31) from (34) for the tangent portfolio, and by deducting (43) and (45) from (48) for the constrained portfolio.

d) Loss due to the use of the two estimators $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$

Finally, we notice that even though $\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ increases with the Sharpe ratio –as shown in (18) and (48) for the tangent and the constrained portfolios, respectively– $\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ decreases. Again, since the maximum utility increases with the Sharpe ratio, this leads to a reduction in the relative percentage loss. The bottom line is that the Sharpe ratio has a positive effect on $\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ as it reduces it. This is consistent with the modern portfolio theory since the Sharpe ratio is an important indicator used in the mean-variance analysis framework.

3.2.3. Number of assets in the portfolios

Including more assets in the risky portfolio increases the chance for estimation error in $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$, increasing both $\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ and $\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$, which leads to a performance loss. However, in real life, increasing *n* can also improve the Sharpe ratio which improves the maximum utility of the portfolio. If the newly included assets are good enough, this improved Sharpe ratio can actually lead to a reduction in $\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$, compensating the increment of $\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$. But even in the case where both $\rho\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ and $\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ increase (which is not what the investor wants) with the addition of new assets, these higher expected absolute and relative loss can be offset by the higher maximum utility made possible by the improved Sharpe ratio. At the end this can lead to a greater expected utility despite a worse $\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$ and $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$. In other words, the expected utility of a given portfolio can be higher than the one of another portfolio that has both a lower $\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$ and $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$. This means that an investor should only include more risky assets in his portfolio if it increases its Sharpe ratio.

In addition to the explanations involving the equations related to the portfolios, Table 3.1 illustrates some of the analysis that have been explained about the tangent portfolios. It shows the relative performance loss in tangent portfolios due to the different types of estimation errors –using $\widehat{\boldsymbol{\mu}}$ or $\widehat{\boldsymbol{\Sigma}}$ or both– and its evolution relatively to different variables: number of assets and amount of information available. There are two panels, on the left, Panel A corresponds to portfolios with a Sharpe ratio of 0.2, while Panel B shows portfolios with a Sharpe ratio of 0.4.

Table 3.1. $\%L_p(w_p, \hat{w}_p)$**: percentage (%) loss of expected utility in tangent portfolios due to estimation errors in the means and covariance matrix of returns.**

Panel A

n	t	$\%L_T(\boldsymbol{w}_T, \hat{\boldsymbol{w}}_T \vert \boldsymbol{\Sigma})$	$\%L_T(\boldsymbol{w}_T, \hat{\boldsymbol{w}}_T \vert \boldsymbol{\mu})$	Interaction	$\%L_T(\boldsymbol{w}_T, \hat{\boldsymbol{w}}_T)$
$\theta = 0.2$					
1	60	41.67	4.31	6.18	52.15
	120	20.83	1.90	1.46	24.19
	240	10.42	0.89	0.36	11.66
	360	6.94	0.58	0.16	7.68
	480	5.21	0.43	0.09	5.73
2	60	83.33	6.85	17.61	107.80
	120	41.67	2.93	4.09	48.69
	240	20.83	1.35	0.99	23.17
	360	13.89	0.88	0.43	15.20
	480	10.42	0.65	0.24	11.31
5	60	208.33	16.64	89.69	314.66
	120	104.17	6.44	19.62	130.23
	240	52.08	2.84	4.61	59.53
	360	34.72	1.81	2.01	38.54
	480	26.04	1.33	1.12	28.49
10	60	416.67	42.99	387.46	847.12
	120	208.33	13.95	75.36	297.64
	240	104.17	5.65	16.85	126.67
	360	69.44	3.51	7.23	80.19
	480	52.08	2.54	4.00	58.62
25	60	1,041.67	336.67	5,211.57	6,589.91
	120	520.83	55.53	591.64	1,168.01
	240	260.42	17.18	110.77	388.37
	360	173.61	9.81	45.19	228.61
	480	130.21	6.81	24.39	161.42

Panel B

n	t	$\%L_T(\boldsymbol{w}_T, \hat{\boldsymbol{w}}_T \vert \boldsymbol{\Sigma})$	$\%L_T(\boldsymbol{w}_T, \hat{\boldsymbol{w}}_T \vert \boldsymbol{\mu})$	Interaction	$\%L_T(\boldsymbol{w}_T, \hat{\boldsymbol{w}}_T)$
$\theta = 0.4$					
1	60	10.42	4.31	1.55	16.27
	120	5.21	1.90	0.37	7.47
	240	2.60	0.89	0.09	3.58
	360	1.74	0.58	0.04	2.36
	480	1.30	0.43	0.02	1.75
2	60	20.83	6.85	4.40	32.09
	120	10.42	2.93	1.02	14.37
	240	5.21	1.35	0.25	6.81
	360	3.47	0.88	0.11	4.46
	480	2.60	0.65	0.06	3.32
5	60	52.08	16.64	22.42	91.14
	120	26.04	6.44	4.90	37.39
	240	13.02	2.84	1.15	17.01
	360	8.68	1.81	0.50	11.00
	480	6.51	1.33	0.28	8.12
10	60	104.17	42.99	96.86	244.02
	120	52.08	13.95	18.84	84.87
	240	26.04	5.65	4.21	35.91
	360	17.36	3.51	1.81	22.68
	480	13.02	2.54	1.00	16.56
25	60	260.42	336.67	1,302.89	1,899.98
	120	130.21	55.53	147.91	333.65
	240	65.10	17.18	27.69	109.98
	360	43.40	9.81	11.30	64.51
	480	32.55	6.81	6.10	45.47

Source: authors' elaboration, based on TABLE 1 presented by Kan and Zhou (2007)

It is observed that the amount of information has a positive effect on the performance relative loss because it is reduced as t increases. As shown in the 3rd column of each panel in Table 3.1, Panel B's $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\Sigma})$ is always lower than that of Panel A –in which the Sharpe ratio is lower. By comparing the 4th column of both panels, it can be observed that $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\mu})$ remains the same whether $\theta = 0.2$ or $\theta = 0.4$. As shown in "Interaction" in the 5th column –which is calculated by deducting the 3rd and 4th column from the 6th column–, the relative performance loss due to the interactive effect from using both estimators decreases as the Sharpe ratio increases. Finally, by comparing the two panels we can see that $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T)$ decreases when the Sharpe ratio of the portfolios is higher.

For further clarification, we will now only focus on the expected relative loss $\%\boldsymbol{L_p}(\boldsymbol{w_p}, \widehat{\boldsymbol{w}}_{\boldsymbol{p}})$ since this is what determines the final performance of the portfolio. We will only discuss the expected absolute loss if it helps to understand the behavior of the portfolios.

4. COMPARING THE PORTFOLIOS' PERFORMANCES

In this section, based on our numerical experiences' results, we compare the percentage loss of expected out-of-sample performance due to estimation error of sample tangent portfolios with the one of sample constrained portfolios. Later, we compare their expected utility. To perform this analysis, we used 14 data sets to create 14 sample tangent portfolios and 14 sample constrained portfolios that we compared to each other.

<u>Composition of the Data Sets (DS)</u> [1]:

-DS1 is composed of 6 portfolios formed on size and book-to-market. It contains 1,110 monthly returns and starts in July 1926.

-DS2 is composed of 6 portfolios formed on size and momentum. It contains 1,104 monthly returns and starts in January 1927.

[1] All data can be found in the following data library: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

-DS3 is composed of 10 industry-based portfolios. It contains 1,110 monthly returns and starts in July 1926.

-DS4 is composed of 24 portfolios formed on size and momentum, originally composed of 25 portfolios but one was deleted because data was missing. It contains 1,104 monthly returns and starts in January 1927.

-DS5 is composed of 25 portfolios formed on size and book-to-market ratio. It contains 1,110 monthly returns and starts in July 1926.

-DS6 is composed of 25 portfolios formed on book-to-market ratio and operating profitability. It contains 666 monthly returns and starts in July 1963.

-DS7 is composed of 25 portfolios formed on size and investment. It contains 666 monthly returns and starts in July 1963.

-DS8 is composed of 25 portfolios formed on size and operating profitability. It contains 666 monthly returns and starts in July 1963

-DS9 is composed of 30 industry-based portfolios. It contains 1,110 monthly returns and starts in July 1926.

-DS10 is composed of 32 portfolios formed on size, book-to-market ratio and investment. It contains 666 monthly returns and starts in July 1963.

-DS11 is composed of 32 portfolios formed on size, book-to-market ratio and operating profitability. It contains 666 monthly returns and starts in July 1963.

-DS12 is composed of 32 portfolios formed on size, operating profitability and investment. It contains 666 monthly returns and starts in July 1963.

-DS13 is composed of 40 industry-based portfolios, originally composed of 49 portfolios but 9 were deleted because data was missing. It contains 1,110 monthly returns and starts in July 1926.

-DS14 is composed of 70 portfolios formed on size and book-to-market ratio, originally composed of 100 portfolios but 30 were deleted because data was missing. It contains 1,110 monthly returns and starts in July 1926.

As explained above, the tangent and constrained portfolios were elaborated with the exact same sets of assets, so that the only difference between these portfolios is the resources allocated to them, their weights vector. The comparison between tangent and constrained portfolios has been made for different values of t periods of monthly historical returns: with t=60, t=120, t=240 and t=360 and t=480. The portfolios' composition goes from n=6 to 70 assets. Each asset is a portfolio composed of all the

NYSE, AMEX and NASDAQ stocks and is formed on various criteria such as size (market equity), book-to-market equity ratio, operating profitability (operating profit divided by book equity), investment and industry.

For a given set of assets, we define the differential loss as the difference between the loss of expected utility using a tangent portfolio and the loss of expected utility when using a constrained portfolio (with a given number of constraints).

a) When $\boldsymbol{\Sigma}$ is known:

$$\Delta\%(T, Q|\boldsymbol{\Sigma}) = \%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\Sigma}) - \%L_Q\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q\middle|\boldsymbol{\Sigma}\right), \tag{56}$$

where $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\Sigma})$ and $\%L_Q\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q\middle|\boldsymbol{\Sigma}\right)$ are the expressions of (52) using a tangent and a constrained portfolio, respectively.

b) When $\boldsymbol{\mu}$ is known:

$$\Delta\%(T, Q|\boldsymbol{\mu}) = \%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\mu}) - \%L_Q\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q\middle|\boldsymbol{\mu}\right), \tag{57}$$

where $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T|\boldsymbol{\mu})$ and $\%L_Q\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q\middle|\boldsymbol{\mu}\right)$ are the expressions of (53) using a tangent and a constrained portfolio, respectively.

c) When none of the parameters $\boldsymbol{\mu}$ or $\boldsymbol{\Sigma}$ is known:

$$\Delta\%(T, Q) = \%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T) - \%L_Q\left(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q\right), \tag{58}$$

where $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T)$ and $\%L_Q(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q)$ are the expressions of (54) respectively using a tangent and a constrained portfolio. Equation (58) is the differential loss between a tangent and a constrained portfolio for a given set of assets. For example, $\Delta\%(T, Q) > 0$ signifies that $\%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T)$ is higher than $\%L_Q(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q)$, meaning that the constrained portfolio is superior to the tangent one in term of relative loss-performance, and vice versa.

4.1 Comparing the tangent and the constrained portfolio percentage loss

In this section, we compared the performance percentage loss of the tangent and constrained portfolios using our 14 sets of assets. To ease the comparison between both portfolios, the sample constrained portfolios were elaborated with parameters $m = 1$ and $\boldsymbol{b} = 1$ as in the orthogonal portfolio presented in Section *2.2*. For the reasons explained in Section *2.3.2*, the findings obtained using these parameters also hold with other parameters and therefore with any constrained portfolio Q of the type presented in Section *2.3*.

a) $\Delta\%(T, Q|\boldsymbol{\Sigma})$

The results concerning the differential loss due to the use of $\widehat{\boldsymbol{\mu}}$ are not interesting because the expressions of the relative loss knowing $\boldsymbol{\Sigma}$ are very similar in both the tangency and constrained portfolios, as we can observe in (13) for the tangency portfolio and in (43) for the constrained portfolios.

b) $\Delta\%(T, Q|\boldsymbol{\mu})$

When we compared the performance loss of the 14 tangent portfolios with that of the 14 constrained ones, we noticed that $\Delta\%(T, Q|\boldsymbol{\mu})$ is always positive. It seems to indicate that constrained portfolios give better results than the tangent portfolios when it comes to performance loss due to the use of an estimator of the covariance matrix $\boldsymbol{\Sigma}$. Also, all other things being equal, the differential loss of performance due to the use of $\widehat{\boldsymbol{\Sigma}}$ seems to increase with the number of assets n in the portfolio, and it tends to zero as the information available t increases. The following tables report the results obtained concerning $\Delta\%(T, Q|\boldsymbol{\mu})$ and its impact on the expected utility for the 14 data sets.

Table 4.1. $\boldsymbol{\Delta\%(T, Q|\mu)}$: differential loss (%) between 14 tangent and constrained portfolios due to the use of an estimator of $\boldsymbol{\Sigma}$ as a function of the number of assets *n* and the amount of information available *t*.

$\Delta\%(T, Q|\boldsymbol{\mu})$

DS / *t*	DS1 $n=6$	DS2 $n=6$	DS3 $n=10$	DS4 $n=24$	DS5 $n=25$	DS6 $n=25$	DS7 $n=25$	DS8 $n=25$	DS9 $n=30$	DS10 $n=32$	DS11 $n=32$	DS12 $n=32$	DS13 $n=40$	DS14 $n=70$
60	5.20	7.43	24.82	98.39	110.34	88.18	91.16	141.92	427.99	258.45	255.19	212.42	1946.75	-
120	1.48	2.38	6.73	13.45	14.03	11.13	12.29	20.27	44.46	20.91	20.21	16.10	78.06	243.53
240	0.53	0.93	2.39	3.58	3.61	2.93	3.44	5.75	11.09	5.02	4.75	3.78	15.76	15.20
360	0.31	0.57	1.41	1.91	1.90	1.56	1.90	3.17	5.83	2.64	2.47	1.97	7.82	5.49
480	0.22	0.41	0.99	1.28	1.26	1.05	1.29	2.16	3.88	1.76	1.64	1.31	5.05	3.00

Source: authors' elaboration

Table 4.1 shows that due to the properties of the constrained portfolios, $\Delta\%(T, Q|\boldsymbol{\mu})$ is always positive, regardless of the amount of information available or the number of assets in the portfolios. We saw in Section *3.2.2* that the orthogonality and the use of the minimum-variance portfolio in the constrained portfolios allow them, by design, to have a lower estimation error in the covariance matrix $\boldsymbol{\Sigma}$ compared to the tangent portfolios. This explains why the performance loss due to the use of $\widehat{\boldsymbol{\Sigma}}$ is better when using a constrained portfolio rather than a tangent one, as Table 4.1 supports. This reduced performance loss can allow the constrained portfolio to reach a higher expected utility than the tangent one. The following two tables show the expected utility obtained for the tangent and the constrained portfolios when $\boldsymbol{\mu}$ is known.

For reasons of readability, through this paper, the expected utility will always be multiplied by 100 in the tables.

Table 4.2 $E[U(\widehat{\boldsymbol{w}}_T|\boldsymbol{\mu})]$: expected utility of the tangency portfolios knowing $\boldsymbol{\mu}$ depending on the number of assets n and the amount of information available t.

$\mathbb{E}[U(\widehat{\boldsymbol{w}}_T)|\boldsymbol{\mu}]$

t \ DS	DS1 n=6	DS2 n=6	DS3 n=10	DS4 n=24	DS5 n=25	DS6 n=25	DS7 n=25	DS8 n=25	DS9 n=30	DS10 n=32	DS11 n=32	DS12 n=32	DS13 n=40	DS14 n=70
60	0.98	1.03	0.36	-4.36	-4.78	-6.93	-13.68	-4.58	-6.52	-39.25	-27.60	-51.35	-46.49	-
120	1.15	1.20	0.54	1.08	0.90	1.30	2.57	0.86	0.24	0.44	0.31	0.57	-0.81	-25.28
240	1.20	1.25	0.59	1.87	1.67	2.42	4.79	1.60	0.90	3.81	2.68	4.98	0.96	-0.12
360	1.22	1.27	0.61	2.03	1.82	2.64	5.21	1.75	1.02	4.37	3.07	5.71	1.21	1.30
480	1.22	1.28	0.61	2.09	1.88	2.73	5.39	1.80	1.06	4.58	3.22	6.00	1.30	1.72

Source: authors' elaboration

The expected utility of the tangent portfolio shown in Table 4.2 is used as a benchmark against which the performances of the constrained portfolios are compared. The cells in which the expected utility of the tangency portfolio is higher than that of the constrained portfolios have been highlighted in grey.

Table 4.3. $E\left[U\left(\widehat{\boldsymbol{w}}_Q|\boldsymbol{\mu}\right)\right]$: expected utility of the constrained portfolios knowing $\boldsymbol{\mu}$ depending on the number of assets n and the amount of information available t.

$\mathbb{E}\left[U\left(\widehat{\boldsymbol{w}}_Q\right)|\boldsymbol{\mu}\right]$

t \ DS	DS1 n=6	DS2 n=6	DS3 n=10	DS4 n=24	DS5 n=25	DS6 n=25	DS7 n=25	DS8 n=25	DS9 n=30	DS10 n=32	DS11 n=32	DS12 n=32	DS13 n=40	DS14 n=70
60	1.04	1.12	0.50	-2.11	-2.50	-4.22	-7.25	-1.56	-1.35	-23.07	-17.18	-33.90	-15.20	-
120	1.16	1.22	0.57	1.35	1.16	1.58	2.83	1.06	0.66	1.32	0.95	1.49	0.29	-19.43
240	1.20	1.26	0.59	1.90	1.71	2.44	4.30	1.45	0.91	3.57	2.64	4.75	1.02	0.24
360	1.21	1.27	0.60	2.01	1.82	2.61	4.59	1.53	0.95	3.96	2.93	5.31	1.13	1.42
480	1.22	1.28	0.60	2.06	1.87	2.68	4.70	1.57	0.97	4.11	3.04	5.52	1.17	1.78

Source: authors' elaboration

When comparing Table 4.3 with Table 4.2, it can be observed that the expected utility of the constrained portfolios is always higher than the one of the tangent portfolios when $t \leq 120$ and knowing $\boldsymbol{\mu}$. When more information is available, the expected utility of the tangent portfolios tends to be higher than the one of the constrained portfolios. It means that the constrained portfolios' ability to reduce estimation error when using an estimator of $\boldsymbol{\Sigma}$ is more significant when the information available is lower. The cells in which the expected utility of the tangency portfolio is higher than that of the constrained portfolios have been highlighted in grey.

c) $\Delta\%(T, Q)$

Constrained portfolios seem to lead to a lower relative loss of performance than the tangent ones when the number of assets composing them is reduced. $\Delta\%(T, Q)$ is positive when *n* is lower but is more likely to be negative as *n* increases. The threshold seems to be around *n*=25. This can be explained by the fact that a higher *n* allows the two portfolios to reach a higher Sharpe ratio, leading to a higher maximum utility in both; however, the tangent portfolio is entirely oriented into Sharpe ratio maximization, as opposed to the constrained portfolio that allocates a part of its resources into the minimum-variance portfolio G that enters its composition. This signifies that when *n* is already high, –which means that the portfolio is already highly diversified– the marginal reduction in variability of Q due to the increased diversification is lower than the marginal increase in return. This makes the minimum-variance portfolio G used in the constrained portfolio redundant, dragging its Sharpe ratio down and making its maximum utility lower than the one of the tangent portfolio. As a result, $\rho(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$ is distributed among a higher maximum utility with the tangent portfolio, in which the Sharpe ratio and maximum utility increase faster than in the constrained portfolio when assets are added in the portfolios' set; hence the obtention of a higher $\%L_p\left(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p\right)$ in the constrained portfolio than in the tangent one when *n* is already high. To put it simply, $\Delta\%(T, Q)$ is sensitive to the relation between the maximum utility of the constrained portfolio and the one of the tangent portfolio. Based on our results, it seems that $\Delta\%(T, Q)$ is negative –indicating that the relative performance loss of the constrained portfolio is worse than the one of the tangent portfolio– when the maximum utility $U\left(\boldsymbol{w}_Q\right)$ of the constrained portfolio represents around 85% or less of the maximum utility $U(\boldsymbol{w}_T)$ of the tangent portfolio. The results of our numerical experiments on $\Delta\%(T,Q)$ are presented in Table 4.4. The top part shows the differential loss between the two portfolios, the bottom part shows the characteristics of the portfolios: their Sharpe ratio, their maximum –potential– utility and relation between them.

Table 4.4. $\boldsymbol{\Delta\%(T,Q)}$: differential loss (%) between 14 tangent and constrained portfolios due to the use of both estimators $\boldsymbol{\hat{\mu}}$ and $\boldsymbol{\hat{\Sigma}}$ and depending on the number of assets *n* and the amount of information available *t*.

$\Delta\%(T,Q)$

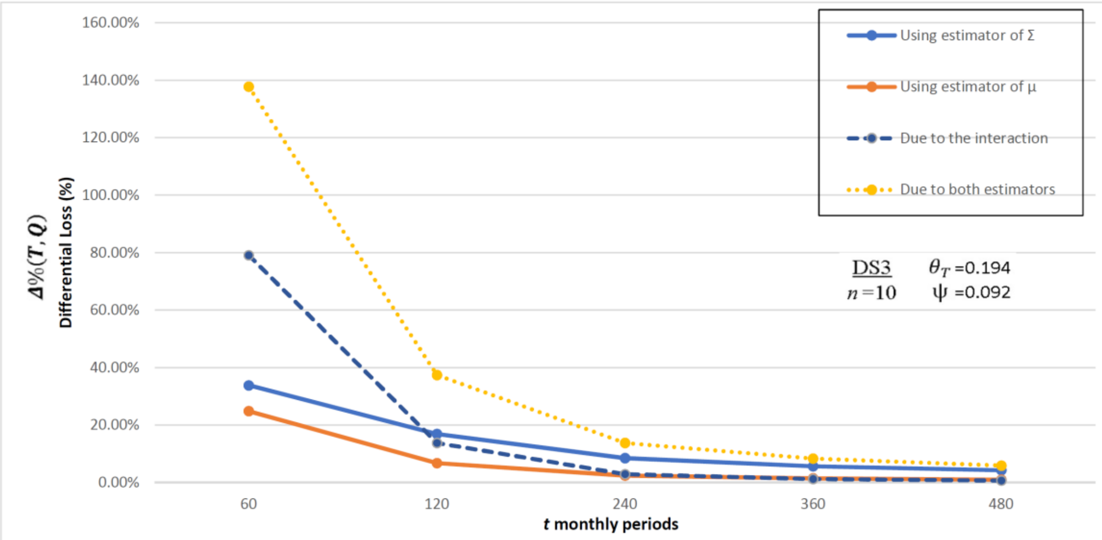
t \ DS	DS1 n=6	DS2 n=6	DS3 n=10	DS4 n=24	DS5 n=25	DS6 n=25	DS7 n=25	DS8 n=25	DS9 n=30	DS10 n=32	DS11 n=32	DS12 n=32	DS13 n=40	DS14 n=70
60	47.72	48.38	137.71	258.81	324.73	227.05	78.32	67.53	495.25	299.85	465.81	294.53	2967.60	-
120	16.58	16.93	37.36	27.65	32.38	22.09	2.15	-16.78	-14.46	5.94	16.86	10.62	-52.61	690.81
240	6.93	7.10	13.76	6.60	7.60	5.10	-0.78	-9.17	-12.58	-0.85	1.74	1.06	-24.81	31.26
360	4.35	4.47	8.28	3.40	3.90	2.60	-0.74	-6.05	-8.66	-0.93	0.45	0.26	-15.40	10.73
480	3.17	3.26	5.90	2.23	2.55	1.70	-0.61	-4.50	-6.54	-0.80	0.13	0.06	-11.12	5.77
Characteristics of the portfolios														
θ_T	0.273	0.279	0.194	0.366	0.348	0.419	0.589	0.341	0.264	0.550	0.461	0.629	0.298	0.376
ψ	0.234	0.221	0.092	0.307	0.294	0.374	0.496	0.250	0.148	0.464	0.400	0.560	0.188	0.344
$U(w_T)$	1.24	1.30	0.63	2.23	2.02	2.93	5.78	1.94	1.16	5.05	3.55	6.60	1.48	2.36
$U(w_Q)$	1.24	1.29	0.61	2.17	1.98	2.84	4.98	1.64	1.02	4.44	3.29	6.00	1.26	2.35
$\frac{U(w_Q)}{U(w_T)}$	99.5%	99.6%	97.5%	97.3%	97.8%	97.2%	86.2%	84.8%	87.7%	88.0%	92.6%	90.8%	85.6%	99.5%

Source: authors' elaboration

All cases in which $\Delta\%(T,Q)$ are negative are highlighted in grey. As stated above, we notice that these cases tend to happen when the number of assets is high ($n \geq 25$), and more specifically when the maximum utility of the constrained portfolios represents 88% or less of the maximum utility of the tangent portfolios. This is depicted in the last row of Table 4.4. As seen in the second row of Table 4.4, the constrained portfolio seems to always lead to an overall lower $\%L_p(\boldsymbol{w}_p, \hat{\boldsymbol{w}}_p)$ than the tangency portfolio when the number of periods of historical returns is small ($t \leq 60$). The lower the information amount, the higher the estimation error risk, and as the constrained portfolio is designed to mitigate estimation errors, it follows that this portfolio does better than the tangent one in a situation of high estimation error.

On the contrary, and as seen before, the estimators become more precise as t increases. This means that estimation error tends to 0 when $t \to \infty$. This fact holds for both the sample tangent and the sample constrained portfolios. The consequence is that the differential loss using any estimator also tends to 0 as t increases. It applies whether the estimation error comes from using $\hat{\boldsymbol{\mu}}$ or $\hat{\boldsymbol{\Sigma}}$ or both. On Figure 4.1 we clearly see that the relative performance losses of both portfolios converge and tend to 0 as the amount of information increases.

Figure 4.1. $\Delta\%(\boldsymbol{T},\boldsymbol{Q})$: differential loss (%) between the relative performance loss of a tangency portfolio and that of a constrained portfolio due to estimation error in the parameters with a given set of assets (DS3), *n*=10.



Source: authors' elaboration

In Figure 4.1, we can note that the differential loss is always positive, even with as much information as *t*=480. This signifies that the relative performance loss of the constrained portfolio is better than the one of the tangency portfolio, despite the fact that the latter has a Sharpe ratio more than twice as high as the one of the former. This suggests that the constrained portfolio is very good at mitigating the impact of estimation error, or at least better than the tangent portfolio in this respect. We note that this occurs when little information is available and that the amount of assets is limited below a certain threshold.

As shown in the following tables, the fact that $\Delta\%(T,Q)$ is positive –meaning that $\%L_Q\left(\boldsymbol{w}_Q,\hat{\boldsymbol{w}}_Q\right)$ is lower than $\%L_T(\boldsymbol{w}_T,\hat{\boldsymbol{w}}_T)$– can be significant enough to allow the constrained portfolios to reach a higher expected utility than the tangent ones. This is notably the case when $t \leq 120$ and when $n \leq 24$.

Table 4.5. $E[U(\widehat{w}_T)]$: expected utility of the tangent portfolios using both estimators $\widehat{\mu}$ and $\widehat{\Sigma}$, depending on the number of assets n and the amount of information available t.

$\mathbb{E}[U(\widehat{w}_T)]$

t \ DS	DS1 n=6	DS2 n=6	DS3 n=10	DS4 n=24	DS5 n=25	DS6 n=25	DS7 n=25	DS8 n=25	DS9 n=30	DS10 n=32	DS11 n=32	DS12 n=32	DS13 n=40	DS14 n=70
60	-1.54	-1.50	-5.00	-40.92	-46.47	-48.61	-55.37	-46.27	-88.94	-149.41	-137.77	-161.51	-470.47	-
120	0.13	0.18	-1.35	-5.81	-6.52	-6.12	-4.85	-6.56	-10.27	-11.58	-11.71	-11.44	-20.97	-177.97
240	0.74	0.79	-0.21	-0.47	-0.80	-0.05	2.31	-0.87	-2.29	0.30	-0.83	1.48	-3.97	-14.26
360	0.92	0.97	0.09	0.64	0.36	1.18	3.75	0.29	-0.81	2.38	1.08	3.72	-1.47	-5.02
480	1.00	1.06	0.24	1.11	0.85	1.70	4.35	0.77	-0.21	3.20	1.84	4.61	-0.53	-2.23

Source: authors' elaboration

The expected utility of the tangent portfolios in Table 4.5 is the benchmark against which the performances of the constrained portfolio are compared. The cells highlighted in grey show the cases in which the expected utility of the tangency portfolio is higher than that of the constrained portfolios.

Table 4.6. $E\left[U(\widehat{w}_Q)\right]$: expected utility of the constrained portfolios using both estimators $\widehat{\mu}$ and $\widehat{\Sigma}$, depending on the number of assets n and the amount of information available t.

$\mathbb{E}\left[U(\widehat{w}_Q)\right]$

t \ DS	DS1 n=6	DS2 n=6	DS3 n=10	DS4 n=24	DS5 n=25	DS6 n=25	DS7 n=25	DS8 n=25	DS9 n=30	DS10 n=32	DS11 n=32	DS12 n=32	DS13 n=40	DS14 n=70
60	-0.94	-0.87	-4.03	-34.19	-39.05	-40.78	-43.80	-38.11	-72.96	-118.20	-112.31	-129.03	-365.36	-
120	0.33	0.40	-1.09	-5.05	-5.74	-5.31	-4.07	-5.83	-9.16	-9.93	-10.29	-9.75	-18.62	-160.85
240	0.82	0.88	-0.12	-0.32	-0.63	0.10	1.95	-0.89	-2.13	0.23	-0.71	1.41	-3.72	-13.45
360	0.97	1.03	0.14	0.69	0.43	1.22	3.20	0.14	-0.80	2.05	1.02	3.40	-1.46	-4.74
480	1.04	1.09	0.27	1.12	0.88	1.70	3.72	0.58	-0.25	2.78	1.71	4.19	-0.59	-2.09

Source: authors' elaboration

As explained earlier, and now by comparing Table 4.5 and Table 4.6, we observe that the estimation error mitigation profile of the constrained portfolio appears to result in a greater expected utility than that of the tangent one when $t \leq 120$ and/or when $n \leq 24$. The cells highlighted in grey show the cases in which the expected utility of the tangency portfolio is higher than that of the constrained portfolios.

4.2. Constrained vs tangent portfolios: adding constraints

Adding constraints to a given portfolio reduces its maximum utility, so that in theory if we knew all the parameters it would be counterproductive to do so. But, if we are in the presence of estimation error, as is the case in real life, the impact of estimation error could be reduced by adding constraints to the portfolio.

As explained in Section *2.3.1*, a portfolio can be constrained by using a matrix **A** that contains the coefficients of the constraints and with an m-vector **b**. As a reminder, *m* is the number of constraints to which we subject the portfolios.

The following example illustrates how we conducted the present experiment. In matric **A**, we want the sum of each row to be $\frac{n}{m}$ and the sum of each column to be 1. Additionally, for simplicity, the coefficients of the constraints are either unit entries or zeros and the portfolios can only be subject to *m* constraints if $\frac{n}{m}$ is an integer. Also, as shown in the following example, the unit entries have been arbitrarily added one after another and each vector **b** row's value is $1/m$, so that the total sum of **b**'s rows is 1.[2]

In this example, we have *m*=3 and *n*=6 which gives us

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Whenever possible, we subjected the 14 portfolios introduced in section *4.1* to this constraining method, with *m*=1 (as used in *4.1*), *m*=2, *m*=3, *m*=4, *m*=5, *m*=6 and *m*=8 constraints.

The results appear in the following tables, they show the relative performance loss of the tangent and the different constrained portfolios depending on their number of constraints and assets. Each table corresponds to a given level of information, Table 4.7 shows the results for *t*=120 and Table 4.8 for *t*=480. In Table 4.7 and Table 4.8, the relative performance loss of the tangent portfolio (3rd column) is our reference point. Cells highlighted in grey show cases in which the relative performance loss of the constrained portfolios is higher than the one of the tangent portfolios.

[2] The method used to add constraints to the portfolios is arbitrary and used for simplicity. There are other ways to constrain the portfolios.

Table 4.7. $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$: relative performance loss (%) of tangent and constrained portfolios depending on their number of constraints and assets for a given level of information *t*=120.

$\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$

DS#	n	T	m=1	m=2	m=3	m=4	m=5	m=6	m=8
DS1	n=6	89.63	73.05	57.19	58.91	-	-	-	-
DS2	n=6	86.25	69.32	60.06	40.20	-	-	-	-
DS3	n=10	315.03	277.67	241.34	-	-	144.03	-	-
DS4	n=24	360.14	332.49	308.81	300.64	274.56	-	246.57	210.89
DS5	n=25	422.82	390.44	-	-	-	298.17	-	-
DS6	n=25	308.98	286.89	-	-	-	306.71	-	-
DS7	n=25	183.86	181.70	-	-	-	134.37	-	-
DS8	n=25	438.43	455.21	-	-	-	370.87	-	-
DS9	n=30	983.28	997.74	950.36	883.69	-	773.53	756.19	-
DS10	n=32	329.41	323.47	305.75	-	275.25	-	-	241.39
DS11	n=32	429.86	413.00	387.26	-	374.62	-	-	471.28
DS12	n=32	273.31	262.69	249.76	-	224.84	-	-	173.47
DS13	n=40	1520.48	1573.10	1477.58	-	1305.57	1236.60	-	1023.66
DS14	n=70	7637.10	6946.28	6494.36	-	-	5244.13	-	-

Source: authors' elaboration

Table 4.7 shows that the constrained portfolios are almost always better at mitigating the performance loss due to estimation error than the tangent ones. Furthermore, for $t = 120$, there is always at least one constrained portfolio with m constraints so that $\%L_Q(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q) < \%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T)$.

Table 4.8. $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$: relative performance loss (%) of tangent and constrained portfolios depending on their number of constraints and assets for a given level of information *t*=480.

$\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$

DS#	n	T	m=1	m=2	m=3	m=4	m=5	m=6	m=8
DS1	n=6	19.16	16.00	12.78	13.44	-	-	-	-
DS2	n=6	18.44	15.18	13.45	9.17	-	-	-	-
DS3	n=10	62.06	56.17	49.85	-	-	31.64	-	-
DS4	n=24	50.50	48.27	45.99	45.90	42.98	-	40.60	36.38
DS5	n=25	57.86	55.30	-	-	-	46.79	-	-
DS6	n=25	42.04	40.34	-	-	-	47.93	-	-
DS7	n=25	24.65	25.26	-	-	-	20.71	-	-
DS8	n=25	60.03	64.53	-	-	-	58.22	-	-
DS9	n=30	118.41	124.95	122.27	116.79	-	107.70	108.05	-
DS10	n=32	36.57	37.36	36.35	-	34.60	-	-	33.86
DS11	n=32	48.12	47.99	46.31	-	47.41	-	-	66.74
DS12	n=32	30.11	30.05	29.43	-	28.03	-	-	24.09
DS13	n=40	135.77	146.89	142.33	-	133.69	130.44	-	117.85
DS14	n=70	194.59	188.82	186.28	-	-	175.57	-	-

Source: authors' elaboration

According to our observations, there is always at least one portfolio with *m* constraints that allows us to have $\%L_Q(\boldsymbol{w}_Q, \widehat{\boldsymbol{w}}_Q) < \%L_T(\boldsymbol{w}_T, \widehat{\boldsymbol{w}}_T)$, even for values of t as big as $t = 480$.

We noticed that up to a certain point, the greater *m* is, the more the estimation error tends to be reduced; however, adding too many constraints can also exacerbate the relative performance loss of the portfolios. For example, we can clearly see it in the last column of Table 4.7 and Table 4.8 with DS11. In Table 4.8, adding *m*=8 constraints to the 11th Data Set has the effect to increase the relative performance loss to 66.74%, while the value is at 46.31% when only *m*=2 constraints are added.

Overall, since most cells are not highlighted, results seem to show that –up to a certain point– adding constraints is a very effective way to reduce estimation error. This is in line with our hypothesis that the addition of constraints can have a positive impact on estimation error reduction. This observation does not mean that the expected utility of the constrained portfolios will necessarily be higher than the one of the tangent portfolios, as constraining the portfolio also reduces their maximum utility. But within the 14 data sets, we noticed that for $t \leq 240$, there is always a three-fund portfolio rule with m constraints that outperforms the classic tangent portfolio; not only in terms of estimation error but also regarding its expected utility. This means that for $t \leq 240$, it seems to be worth it to sacrifice some maximum utility by adding constraints.

In Table 4.9, Table 4.10 and Table 4.11, the expected utility of the tangent portfolio (3rd column) is our reference point. Cells highlighted in grey show cases in which the expected utility of the constrained portfolios is lower than the one of the tangent portfolios. Results are shown for the levels or information t=60, t=120 and t=240. As a reminder, we are still using the constraining method explained in Section *4.2*.

Table 4.9. $\mathbb{E}\left[U\left(\hat{w}_p\right)\right]$: expected utility of tangency and constrained portfolios depending on their number of constraints and assets for a given level of information t=60.

DS#	n	E.Ut.T	E.Ut.m=1	E.Ut.m=2	E.Ut.m=3	E.Ut.m=4	E.Ut.m=5	E.Ut.m=8
DS1	n=6	-1.54	-0.94	-0.42	-0.29	-	-	-
DS2	n=6	-1.50	-0.87	-0.45	0.11	-	-	-
DS3	n=10	-5.00	-4.03	-3.28	-	-	-1.44	-
DS4	n=24	-40.92	-34.19	-29.77	-25.93	-22.58	-	-12.39
DS5	n=25	-46.47	-39.05	-	-	-	-22.09	-
DS6	n=25	-48.61	-40.78	-	-	-	-22.69	-
DS7	n=25	-55.37	-43.80	-	-	-	-23.66	-
DS8	n=25	-46.27	-38.11	-	-	-	-21.76	-
DS9	n=30	-88.94	-72.96	-63.39	-55.24	-	-42.05	-
DS10	n=32	-149.41	-118.20	-101.19	-	-74.95	-	-41.07
DS11	n=32	-137.77	-112.31	-96.52	-	-71.32	-	-39.03
DS12	n=32	-161.51	-129.03	-109.60	-	-80.67	-	-44.38
DS13	n=40	-470.47	-365.36	-29.27	-	-20.35	-17.09	-114.13

Source: authors' elaboration

In Table 4.9 the expected utility of the constrained portfolios is always higher that of the tangency portfolios when t=60, no matter the amount of constraints.

Table 4.10. $\mathbb{E}[U(\hat{w}_p)]$: expected utility of tangent and constrained portfolios depending on their number of constraints and assets for a given level of information *t*=120.

DS#	n	E.Ut.T	E.Ut.m=1	E.Ut.m=2	E.Ut.m=3	E.Ut.m=4	E.Ut.m=5	E.Ut.m=8
DS1	n=6	0.13	0.33	0.53	0.35	-	-	-
DS2	n=6	0.18	0.40	0.45	0.76	-	-	-
DS3	n=10	-1.35	-1.09	-0.86	-	-	-0.26	-
DS4	n=24	-5.81	-5.05	-4.53	-4.13	-3.64	-	-2.10
DS5	n=25	-6.52	-5.74	-	-	-	-3.71	-
DS6	n=25	-6.12	-5.31	-	-	-	-3.84	-
DS7	n=25	-4.85	-4.07	-	-	-	-1.67	-
DS8	n=25	-6.56	-5.83	-	-	-	-3.95	-
DS9	n=30	-10.27	-9.16	-8.50	-7.86	-	-6.69	-
DS10	n=32	-11.58	-9.93	-8.98	-	-7.33	-	-4.87
DS11	n=32	-11.71	-10.29	-9.38	-	-7.94	-	-6.02
DS12	n=32	-11.44	-9.75	-8.71	-	-6.93	-	-3.90
DS13	n=40	-20.97	-18.62	-17.40	-	-15.14	-14.12	-11.39
DS14	n=70	-177.97	-160.85	-148.58	-	-	-118.28	-

Source: authors' elaboration

Again, in Table 4.10, we observe that the expected utility of the constrained portfolios is always higher that of the tangent portfolios when *t*=120, no matter the amount of constraints.

Table 4.11. $\mathbb{E}[U(\hat{w}_p)]$: expected utility of tangent and constrained portfolios depending on their number of constraints and assets for a given level of information *t*=240.

DS#	n	E.Ut.T	E.Ut.m=1	E.Ut.m=2	E.Ut.m=3	E.Ut.m=4	E.Ut.m=5	E.Ut.m=8
DS1	n=6	0.74	0.82	0.91	0.62	-	-	-
DS2	n=6	0.79	0.88	0.81	1.04	-	-	-
DS3	n=10	-0.21	-0.12	-0.04	-	-	0.20	-
DS4	n=24	-0.47	-0.32	-0.18	-0.15	0.01	-	0.34
DS5	n=25	-0.80	-0.63	-	-	-	-0.16	-
DS6	n=25	-0.05	0.10	-	-	-	-0.21	-
DS7	n=25	2.31	1.95	-	-	-	2.51	-
DS8	n=25	-0.87	-0.89	-	-	-	-0.51	-
DS9	n=30	-2.29	-2.13	-2.00	-1.85	-	-1.57	-
DS10	n=32	0.30	0.23	0.37	-	0.60	-	0.65
DS11	n=32	-0.83	-0.71	-0.53	-	-0.50	-	-0.96
DS12	n=32	1.48	1.41	1.49	-	1.69	-	2.23
DS13	n=40	-3.97	-3.72	-3.52	-	-3.14	-2.96	-2.45
DS14	n=70	-14.26	-13.45	-12.93	-	-	-11.49	-

Source: authors' elaboration

We can see in Table 4.11 that within our data sets and for *t*=240, there is always at least a constrained portfolio with a given *m* amount of constraints that dominates the tangent portfolio in terms of expected utility.

Below, with Table 4.12, Table 4.13 and Table 4.14, we highlight the fact that despite leading to a lower maximum utility, adding constraints to a portfolio can allow it to achieve a higher expected utility than a tangent portfolio. As a reminder, $U(\boldsymbol{w}_p)$ represents the maximum potential –theoretical– utility that can be reached for a given portfolio *p*.

Table 4.12. $\mathbb{E}[\boldsymbol{U}(\widehat{\boldsymbol{w}}_{\boldsymbol{p}})]$: evolution of the expected and maximum utility of a portfolio with a given set of assets (DS4) as a function of the number of constraints it is subjected to, and dependent on the amount of information *t*.

DS4
n=24

t	T	m=1	m=2	m=3	m=4	m=6	m=8
60	-40.92	-34.19	-29.77	-25.93	-22.58	-16.72	-12.39
120	-5.81	-5.05	-4.53	-4.13	-3.64	-2.84	-2.10
240	-0.47	-0.32	-0.18	-0.15	0.01	0.14	0.34
360	0.64	0.69	0.77	0.73	0.83	0.84	0.94
480	1.11	1.12	1.17	1.11	1.19	1.15	1.20
$U(\boldsymbol{w}_p)$	2.23	2.17	2.17	2.06	2.09	1.94	1.89

Source: authors' elaboration

Table 4.12 clearly shows that adding constraints can indeed increase the expected utility of the portfolio, despite a reduction in the maximum utility as observed in the last row. We also notice that in the specific case shown by the table, the constrained portfolios always yield a higher expected utility than the tangent portfolio, even with as much information as *t*=480. It is interesting to note that in this specific case, the most constrained portfolio (with *m=8*) is also the one that leads to the highest expected utility, while simultaneously having the lowest maximum utility. This means that the constrained portfolio does so well at mitigating the relative performance loss (as suggested by Table 4.7 and Table 4.8) that it does more than compensate for its lower maximum utility.

Table 4.13. $\mathbb{E}\left[U(\widehat{w}_p)\right]$: evolution of the expected and maximum utility of a portfolio with a given set of assets (DS3) as a function of the number of constraints it is subjected to, and dependent on the amount of information *t*.

DS3
n=10

t	T	m=1	m=2	m=5
60	-5.00	-4.03	-3.28	-1.44
120	-1.35	-1.09	-0.86	-0.26
240	-0.21	-0.12	-0.04	0.20
360	0.09	0.14	0.20	0.34
480	0.24	0.27	0.31	0.40
$U(w_p)$	0.63	0.61	0.61	0.59

Source: authors' elaboration

Table 4.13 shows results similar to those presented in Table 4.12. It is interesting to note the variability of the expected utility depending on the amount of constraints. For example, in Table 4.13, for t=360, the expected utility of the constrained portfolio with m=1 is 0.14 while the expected utility of the constrained portfolio with m=5 is 0.34. The only difference between these two constrained portfolios is their amount of constraints, they are entirely responsible for these variations in the expected utility.

Table 4.14. $\mathbb{E}\left[U(\widehat{w}_p)\right]$: evolution of the expected and maximum utility of a portfolio with a given set of assets (DS13) as a function of the number of constraints it is subjected to, and dependent on the amount of information *t*.

DS13
n=40

t	T	m=1	m=2	m=4	m=5	m=8
60	-470.47	-365.36	-29.27	-20.35	-17.09	-114.13
120	-20.97	-18.62	-17.40	-15.14	-14.12	-11.39
240	-3.97	-3.72	-3.52	-3.14	-2.96	-2.45
360	-1.47	-1.46	-1.36	-1.19	-1.11	-0.86
480	-0.53	-0.59	-0.53	-0.42	-0.38	-0.22
$U(w_p)$	1.48	1.26	1.26	1.26	1.24	1.23

Source: authors' elaboration

Table 4.14 shows that the results obtained earlier hold even when n is high. Here again, despite leading to a lower maximum utility, the constrained portfolios can outperform the expected utility of the tangent ones. It is important to note that adding more constraints does not necessarily imply a higher utility; for example, as highlighted in grey in Table 4.14, with *t*=60 the expected utility of the portfolio with *m*=8 constraints is much lower than the expected utility of the portfolio with *m*=2 constraints.

In the appendix, we explained what determines whether a tangent portfolio will have a higher or lower expected utility than that of a constrained portfolio. To summarize, our findings indicate that tangent portfolios rely on their higher maximum performance, which are driven by their higher Sharpe ratio, while the constrained portfolios are based on their better estimation error mitigation profile.

In the next section we tried to determine if the expected utility of the constrained portfolios can be improved by randomizing their constraints.

4.3 Going further: randomizing constrained portfolio vs other portfolio rules

As explained previously, the unit entries used in matrix **A** have been arbitrarily added one after another for simplicity. In this section, we randomized the unit entries in matrix **A** to see if the results could be further improved. The other characteristics of the method used in Section *4.2* to define matrix **A** and vector **b** remain unchanged.

Using MATLAB, the unit entries have randomly been rearranged many times. Each time, for a given amount of constraints, we registered the corresponding performances of the randomized constrained portfolios: their $\%L_p(\boldsymbol{w}_p, \widehat{\boldsymbol{w}}_p)$ and their maximum potential utility, which are affected by **A**. With these performances, we have calculated and reported the expected utility of the portfolios. We repeated the procedure with 4 data sets: DS1 and DS2 (*n*=6), DS3 (*n*=10) and DS4 (*n*=24). Whenever it was possible, we constrained them with *m*=2, *m*=3 and *m*=4 constraints. We could not do the experiment with more than 24 assets because of limiting computing capacity in creating combinations for matrix **A**. The results obtained, however, are significant enough to draw some conclusions.

The following table reports the sample maximum, minimum and average expected utility obtained by randomizing matrix **A** for two different levels of information: with *t*=60 and *t*=120. The expected utility of the corresponding tangent portfolios for the data sets have been added as a benchmark.

Table 4.15. $\mathbb{E}\left[U(\widehat{\boldsymbol{w}}_{\boldsymbol{p}})\right]$: sample maximum, minimum and average expected utility obtained by randomizing the matrix containing the coefficients of the constraints, for two different levels of information.

		DS1 n=6		DS2 n=6		DS3 n=10	DS4 n=24	
		m=2	m=3	m=2	m=3	m=2	m=3	m=4
t=60	Max.E.Ut.	-0.42	-0.02	-0.34	0.11	-3.27	-25.19	-21.82
	Min.E.Ut.	-0.83	-0.29	-0.50	-0.91	-3.38	-26.03	-22.66
	Avg.E.Ut.	-0.56	-0.26	-0.40	-0.08	-3.30	-25.85	-22.42
t=120	Max.E.Ut.	0.53	0.64	0.59	0.76	-0.86	-3.20	-3.52
	Min.E.Ut.	0.09	-0.14	0.43	0.12	-0.97	-5.54	-3.87
	Avg.E.Ut.	0.38	0.40	0.53	0.58	-0.88	-4.07	-3.62
Tangent portfolio as a benchmark								
t=60	$\mathbb{E}[U(\widehat{\boldsymbol{w}}_T)]$	-1.54		-1.50		-5.00	-40.92	
t=120		0.13		0.18		-1.35	-5.81	

Source: authors' elaboration

In Table 4.15, we can observe that the difference between the sample maximum and minimum expected utility can be significant. It is also showed that the sample average utility of the randomized constraints is higher than the expected utility of the equivalent tangent portfolios.

Here, it is important to keep in mind that the difference found in the expected utility is entirely due to the order in which the coefficients of the constraints have been randomly rearranged in matrix **A**. This shows that the expected utility of the constrained portfolios can indeed be improved by randomizing the constrained assets. We can see by comparing the maximum and the minimum utility within each portfolio for a given *m* number of constraints that the order in which the assets are constrained does matter and has a significant impact on the performance of the portfolios. It can be concluded that for each constrained portfolio, there is an optimal amount of constraints and an optimal way to combine them. Such an optimal constraining method would maximize the expected utility of the portfolio and goes even further than the results presented in Section *4.2*, in which the coefficients of the constraints in matrix **A** were simply added one after another.

There are several limitations to the randomization of constraints, however. First, since it is generated randomly, there is no way to ensure that the arrangement of the unit entries (coefficients of the constraints) is optimal. Second, depending on the number of assets and constraints the portfolios are subjected to, we may be limited by memory and computing power given the many possible combinations for any matrix **A**. This means that the best we can do is to generate as many combinations as possible and use the one that allows us to achieve the maximum expected utility.

4.4 Control experiment: equally weighted portfolios vs all the portfolio rules

In this section, we compared the expected utility of the constrained portfolios to that of the equally weighted portfolio, which is constructed by assigning the same weighting to each asset in the portfolio.

When $t \leq 60$ and/or when *n* is high (ranging from 32 to 40 assets, according to our results), meaning that the risk for estimation error is high, an equally weighted portfolio always outperforms all the portfolio rules. Then, all other things being equal, the higher *t*, the less likely the equally weighted portfolio is to be better than the portfolio rules. The equally weighted portfolio is the safest way for an investor to build a portfolio as it does not rely on any estimation. It dramatically reduces its maximum utility, but on the other hand, it will also invariably provide a higher utility than any portfolio rule when little information is available. This method is a very conservative yet efficient way of building a portfolio. In the following table, we show the different expected utilities depending on whether we use of an equally weighted portfolio (3rd column), a tangency portfolio or constrained portfolios.

Table 4.16. $\mathbb{E}\left[U\left(\hat{w}_p\right)\right]$: expected utility for equally weighted portfolios and different constrained portfolios with a given reduced level of information, *t*=60.

DS#	*n*	E.Ut.Eq.w	E.Ut.T	E.Ut.*m* =1	E.Ut.*m* =2	E.Ut.*m* =3	E.Ut.*m* =4	E.Ut.*m* =5	E.Ut.*m* =8
DS1	*n* =6	0.20	-1.54	-0.94	-0.42	-0.29	-	-	-
DS2	*n* =6	0.20	-1.50	-0.87	-0.45	0.11	-	-	-
DS3	*n* =10	0.26	-5.00	-4.03	-3.28	-	-	-1.44	-
DS4	*n* =24	0.21	-40.92	-34.19	-29.77	-25.93	-22.58	-	-12.39
DS5	*n* =25	0.17	-46.47	-39.05	-	-	-	-22.09	-
DS6	*n* =25	0.40	-48.61	-40.78	-	-	-	-22.69	-
DS7	*n* =25	0.35	-55.37	-43.80	-	-	-	-23.66	-
DS8	*n* =25	0.31	-46.27	-38.11	-	-	-	-21.76	-
DS9	*n* =30	0.23	-88.94	-72.96	-63.39	-55.24	-	-42.05	-
DS10	*n* =32	0.42	-149.41	-118.20	-101.19	-	-74.95	-	-41.07
DS11	*n* =32	0.40	-137.77	-112.31	-96.52	-	-71.32	-	-39.03
DS12	*n* =32	0.41	-161.51	-129.03	-109.60	-	-80.67	-	-44.38
DS13	*n* =40	0.23	-470.47	-365.36	-29.27	-	-20.35	-17.09	-114.13

Source: authors' elaboration

Table 4.16 shows that the equally weighted portfolio method largely dominates when little information is available. Except for the case in DS2 with m=3, the expected utility using the tangency and constrained portfolios is always negative while the one of the equally weighted portfolio is always positive.

Table 4.17. $\mathbb{E}\left[U\left(\widehat{w}_p\right)\right]$: expected utility for equally weighted portfolios and different constrained portfolios with a given level of information, t=120.

with t=120

DS#	n	E.Ut.Eq.w	E.Ut.T	E.Ut.m=1	E.Ut.m=2	E.Ut.m=3	E.Ut.m=4	E.Ut.m=5	E.Ut.m=8
DS1	n=6	0.20	0.13	0.33	0.53	0.35	-	-	-
DS2	n=6	0.20	0.18	0.40	0.45	0.76	-	-	-
DS3	n=10	0.26	-1.35	-1.09	-0.86	-	-	-0.26	-
DS4	n=24	0.21	-5.81	-5.05	-4.53	-4.13	-3.64	-	-2.10
DS5	n=25	0.17	-6.52	-5.74	-	-	-	-3.71	-
DS6	n=25	0.40	-6.12	-5.31	-	-	-	-3.84	-
DS7	n=25	0.35	-4.85	-4.07	-	-	-	-1.67	-
DS8	n=25	0.31	-6.56	-5.83	-	-	-	-3.95	-
DS9	n=30	0.23	-10.27	-9.16	-8.50	-7.86	-	-6.69	-
DS10	n=32	0.42	-11.58	-9.93	-8.98	-	-7.33	-	-4.87
DS11	n=32	0.40	-11.71	-10.29	-9.38	-	-7.94	-	-6.02
DS12	n=32	0.41	-11.44	-9.75	-8.71	-	-6.93	-	-3.90
DS13	n=40	0.23	-20.97	-18.62	-17.40	-	-15.14	-14.12	-11.39
DS14	n=70	0.24	-177.97	-160.85	-148.58	-	-	-118.28	-

Source: authors' elaboration

The results showed in Table 4.17 are similar to those of Table 4.16. The difference is that here, in Table 4.17, more information is available. This allows portfolios with fewer assets (n=6) to eventually achieve greater expected utility than that of the equally weighted portfolio. This is no longer the case when data sets are composed of a larger amount of assets. The portfolios that have a higher expected utility than that of the equally weighted portfolio are highlighted in grey.

Table 4.18. $\mathbb{E}[U(\hat{w}_p)]$: expected utility for equally weighted portfolios and different constrained portfolios with a given level of information, *t*=240.

with t=240

DS#	n	E.Ut.Eq.w	E.Ut.T	E.Ut.m=1	E.Ut.m=2	E.Ut.m=3	E.Ut.m=4	E.Ut.m=5	E.Ut.m=8
DS1	n=6	0.20	0.74	0.82	0.91	0.62	-	-	-
DS2	n=6	0.20	0.79	0.88	0.81	1.04	-	-	-
DS3	n=10	0.26	-0.21	-0.12	-0.04	-	-	0.20	-
DS4	n=24	0.21	-0.47	-0.32	-0.18	-0.15	0.01	-	0.34
DS5	n=25	0.17	-0.80	-0.63	-	-	-	-0.16	-
DS6	n=25	0.40	-0.05	0.10	-	-	-	-0.21	-
DS7	n=25	0.35	2.31	1.95	-	-	-	2.51	-
DS8	n=25	0.31	-0.87	-0.89	-	-	-	-0.51	-
DS9	n=30	0.23	-2.29	-2.13	-2.00	-1.85	-	-1.57	-
DS10	n=32	0.42	0.30	0.23	0.37	-	0.60	-	0.65
DS11	n=32	0.40	-0.83	-0.71	-0.53	-	-0.50	-	-0.96
DS12	n=32	0.41	1.48	1.41	1.49	-	1.69	-	2.23
DS13	n=40	0.23	-3.97	-3.72	-3.52	-	-3.14	-2.96	-2.45
DS14	n=70	0.24	-14.26	-13.45	-12.93	-	-	-11.49	-

Source: authors' elaboration

In Table 4.18, even more information is available. This makes the use of the equally weighted portfolio less interesting. The portfolios that have a higher expected utility than that of the equally weighted portfolio are highlighted in grey. When the amount of assets is limited below the threshold around $32 \leq n \leq 40$, the expected utility of the tangent and specifically of the constrained portfolios can be much greater than the one of the equally weighted ones. This means that the equally weighted portfolio is more interesting when little information is available and/or when the portfolios are composed of many assets ($n \geq 32$).

5. CONCLUSION

After analyzing the impact of estimation error, we saw different ways to mitigate it and then compared them to each other. It has been shown that in certain conditions, adding constraints to the portfolios is an effective way to reduce estimation error and even to increase their expected utility, notably when the amount of assets in the portfolios is limited to a certain quantity ($n \leq 24$) and when the amount of information is moderate to high (for $60 < t \leq 480$). To summarize, it can be worth it to sacrifice some maximum utility and to add constraints in order to gain precision in the estimation.

Based on our results, it seems that for an amount of assets around $25 \leq n \leq 36$ and when sufficient information is available –around ($t \geq 360$)– it is preferable to use the classic tangent portfolio method because this method aims for the highest utility. In the case where $60 < t \leq 240$, it seems to be better to use the constrained portfolio method as it reduces estimation error without sacrificing too much maximum utility, leading to a better expected utility. Then, when the estimation risk is too high, namely when $t \leq 60$ or that the quantity of assets in the portfolio exceeds a certain threshold (estimated around $n \geq 36$), the use of an equally weighted portfolio is encouraged as opposed to other portfolio rules that would lead to an excessive estimation error. This would ultimately result in a negative utility.

Table 5.1 summarizes which method to use based on what our observations suggest.

Table 5.1. Most appropriate portfolio construction method based on our observations and focusing on the expected utility.[3]

t \ n	$6 \leq n < 10$	$10 \leq n \leq 24$	$24 < n \leq 36$	$n \geq 36$
$t \leq 60$	Equally weighted	Equally weighted	Equally weighted	Equally weighted
$60 < t \leq 120$	Constrained	Equally weighted	Equally weighted	Equally weighted
$120 < t \leq 240$	Constrained	Constrained	Constrained	Equally weighted
$t \geq 360$	Constrained	Constrained	Tangent	Equally weighted

Source: author's elaboration

Finally, randomization of the matrix used to constrain portfolios seems to be an interesting and promising approach because our results showed that it can lead to even higher expected utility than using arbitrary constraints as conducted in our research.

[3] This table only provides general guidelines; it is not an absolute reference.

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